Topology and its Applications 159 (2012) 2392–2403

Contents lists available at SciVerse ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

The covering dimension invariants

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ARTICLE INFO

MSC:
54F45
54D35

Keywords:
Covering dimension invariants
Compactification
Wallman-type compactification
Proximity
Uniformity
Normal base
Universal space

ABSTRACT

In the present paper three types of covering dimension invariants of a space X are distinguished. Their sets of values are denoted by $\text{d-Sp}_U(X)$, $\text{d-Sp}_W(X)$ and $\text{d-Sp}_\beta(X)$. One of the exhibited relations between them shows that the minimal values of $\text{d-Sp}_U(X)$, $\text{d-Sp}_W(X)$ and $\text{d-Sp}_\beta(X)$ coincide. This minimal value is equal to the dimension invariant mindim defined by Isbell. We show that if X is a locally compact space, then either $\text{d-Sp}_U(X) = [\operatorname{mindim} X, \infty]$, or $\text{d-Sp}_U(X) = \text{d-Sp}_\beta(X) = \{\dim X\}$. If X is not a pseudocompact space, then $[\dim X, \infty] \subset \text{d-Sp}_U(X)$; if X is a Lindelöff non-compact space, then $\text{d-Sp}_U(X) = [\dim X, \infty]$; if X is a separable metrizable non-compact space, then $\text{d-Sp}_W(X) = [\operatorname{mindim} X, \infty]$. Among the properties of covering dimension invariants the generalization of the compactification theorem of Skljarenko is presented. The existence of compact universal spaces in the class of all spaces X with $w(X) \leqslant \tau$ and $\operatorname{mindim} X \leqslant n$ is proved.

1. Introduction and preliminaries

The covering dimension dim is one of the three main dimensions. Though it appeared in the work of H. Lebesgue the formal definition was given by E. Čech. Its modification for Tychonoff spaces was given by M. Katetov. Yu. Smirnov introduced the covering dimension invariants: δ-dimensions. A δ-dimension of a space is equal to the dimension dim of its δ-compactification. Moreover, dimension dim is one of the δ-dimensions. Therefore, these invariants give more dimensional-type information about the space.

Other covering dimension invariants appeared in works of J. Isbell, M. Charalambous, A. Chigogidze, and S. Iliadis. Their common feature is that a given covering dimension invariant is equal to the dimension dim of some concrete compactification. Many classical theorems of dimension theory hold for them. They turned out to be a good tool for the investigation of dimensional properties of spaces.

In Section 2 we classify and give relations between δ-dimension of Smirnov [28], uniform dimension δd of Isbell [17], dimension d by a normal base of Iliadis [15] (see, also, [13]), uniform dimension μ-dim of Charalambous [4] (see, also, [5]), relative dimension d of Chigogidze [8], and the classical dimension dim. Three types of covering dimension invariants of a space X are distinguished. Their sets of values are denoted by $\text{d-Sp}_U(X)$, $\text{d-Sp}_W(X)$, and $\text{d-Sp}_\beta(X)$. $\text{d-Sp}_U(X)$ is the set of dimensions of all compactifications of X, $\text{d-Sp}_W(X)$ is the set of dimensions of all Wallman-type compactifications of X, and $\text{d-Sp}_\beta(X)$ is the set of dimensions of all β-like compactifications of X [19].

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1 Work supported by the Caratheodory Programme of the University of Patras.
2 Supported by Russian ministry of science and education, project RPN 2.1.1.3704. Modern differential geometry, topology and its applications.

0166-8641/$ – see front matter
doi:10.1016/j.topol.2011.09.047

In Section 3 relations between covering dimension invariants are exhibited. It is shown that the minimal values of d-$\mathrm{Sp}_U(X)$, d-$\mathrm{Sp}_W(X)$ and d-$\mathrm{Sp}_\beta(X)$ coincide. This minimal value is equal to the dimension invariant mindim introduced by Isbell [17].

In Section 4 rangers of d-$\mathrm{Sp}_U(X)$, d-$\mathrm{Sp}_W(X)$, and d-$\mathrm{Sp}_\beta(X)$ are investigated. It is proved that if X is a locally compact space, then either d-$\mathrm{Sp}_U(X) = [\operatorname{mindim} X, \infty]$, or d-$\mathrm{Sp}_U(X) =$ d-$\mathrm{Sp}_\beta(X) = \{\dim X\}$. If X is not a pseudocompact space, then $[\dim X, \infty] \subset$ d-$\mathrm{Sp}_U(X)$. If X is a Lindelöff non-compact space, then d-$\mathrm{Sp}_U(X) = [\dim X, \infty]$, and if X is a separable metrizable non-compact space, then d-$\mathrm{Sp}_W(X) = [\operatorname{mindim} X, \infty]$. For the rangers of transfinite dimensions in the class of separable metrizable spaces see [6].

In Section 5 further properties of covering dimension invariants of a space X are presented. The generalization of the compactification theorem of E. Skljarenko [22] is given. The existence of compact universal spaces in the class of all spaces X with $w(X) \leqslant \tau$ and $\operatorname{mindim} X \leqslant n$ is proved.

All spaces under consideration are assumed to be Tychonoff and maps of spaces continuous. Our terminology mostly follows [10,11]. The closure in a space X of a subset A is denoted by $\mathrm{cl}_X A$. By $C^*(X)$ the ring of all real valued bounded continuous functions on space X is denoted.

Let u, v be covers of a space X. The notation $u \succ v$ means that u is a refinement of v. For $u = \{U_\alpha \colon \alpha \in A\}$ and $M \subset X$ we set $M \wedge u = \{U_\alpha \cap M \colon \alpha \in A\}$.

We consider uniform structures on spaces by using families of covers. All uniformities are compatible with the topology of a space. The necessary information about uniformities and proximities can be found in [10,17].

For information about normal bases see [2]. By $\omega(X, \mathcal{F})$ we denote the Wallman-type compactification of X with respect to normal base $\mathcal{F}$, and by $\omega\mathcal{F}$ we denote the normal base on $\omega(X, \mathcal{F})$ which elements are the closures in $\omega(X, \mathcal{F})$ of the elements of $\mathcal{F}$.

We denote by ω the set of all nonnegative integers, by $\mathbb{N}$ the set of all natural numbers, by I the closed interval $[0, 1]$, by $\mathbb{Q}$ the set of rational numbers of I, and by $\mathbb{N}^*$ the set $\beta\mathbb{N} \setminus \mathbb{N}$.

A general approach for defining covering dimension invariants is the following. Let $\mathcal{C}$ be a family of finite open covers of a space X. We say that $\mathcal{C}$-$\dim X \leqslant n$ if and only if any cover $u \in \mathcal{C}$ has a refinement $v \in \mathcal{C}$ with $\operatorname{ord} v \leqslant n$. In the case, where each element from $\mathcal{C}$ has an extension on some fixed compactification bX, it is natural to require the equality $\mathcal{C}$-$\dim X = \dim bX$.

General Theorem *(On the equality $\mathcal{C}$-$\dim X = \dim bX$). Suppose that for a family $\mathcal{C}$ of finite open covers of a space X and a compactification bX of X the following two conditions hold.*

(A) *Every element from $\mathcal{C}$ has an extension on bX (that is, for any $v = \{V_1, \ldots, V_k\} \in \mathcal{C}$ there is an open cover $v' = \{V'_1, \ldots, V'_k\}$ of bX such that $V'_i \cap X = V_i$, $i = 1, \ldots, k$).*
(B) *Every finite open cover of bX has a shrinking which is an extension of some element from $\mathcal{C}$.*

Then, we have

$$\mathcal{C}\text{-}\dim X = \dim bX.$$

Proof. The proof of the inequality $\mathcal{C}$-$\dim X \leqslant \dim bX$. Let $v \in \mathcal{C}$ be an arbitrary cover of X and v' its extension on bX. There is a finite open cover $u' \succ v'$ of bX with $\operatorname{ord} u' \leqslant \dim bX$. By condition (B), without lost of generality, u' may be considered as an extension of $u \in \mathcal{C}$. Hence, $\operatorname{ord} u \leqslant \dim bX$ and $u \succ v$.

The proof of the inequality $\mathcal{C}$-$\dim X \geqslant \dim bX$. Let $v'' = \{V''_i \colon i = 1, \ldots, k\}$ be an arbitrary finite open cover of bX. There is an open shrinking $v' = \{V'_i \colon i = 1, \ldots, k\}$ of v'' such that $\mathrm{cl}_{bX} V'_i \subset V''_i$, $i = 1, \ldots, k$. Without lost of generality, v' may be considered as an extension of $v \in \mathcal{C}$. There is a finite open cover $u = \{U_i \colon i = 1, \ldots, m\} \in \mathcal{C}$ such that $u \succ v$ and $\operatorname{ord} u \leqslant \mathcal{C}$-$\dim X$. Let $u' = \{U'_i \colon i = 1, \ldots, m\}$ be an extension of u on bX. Then, since X is dense in bX, we have

$$\bigcap\bigl\{U'_{i_t} \colon t = 1, \ldots, l\bigr\} \neq \emptyset \quad \text{if and only if} \quad X \cap \bigcap\bigl\{U'_{i_t} \colon t = 1, \ldots, l\bigr\} \neq \emptyset$$

and, therefore,

$$\text{if and only if} \quad \bigcap\{U_{i_t} \colon t = 1, \ldots, l\} \neq \emptyset.$$

Thus, $\operatorname{ord} u' = \operatorname{ord} u \leqslant \mathcal{C}$-$\dim X$. If $U_i \subset V_j \in v$, then $\mathrm{cl}_{bX} U_i \subset \mathrm{cl}_{bX} V_j \subset V''_j$. Besides, $U'_i \subset \mathrm{cl}_{bX} U_i$. Hence, $u' \succ v''$. $\square$

Since all finite open covers of a compact space is the basis of its unique uniformity, the above General Theorem shows that the family $\mathcal{C}$ of the theorem is the basis of that totally bounded uniformity μ on X, the completion of X with respect to which is bX. This observation shows that the choice of a suitable family $\mathcal{C}$ is equivalent to the choice of a basis of some totally bounded uniformity.

2. Covering dimension invariants

Let us remind definitions of covering dimension invariants using the approach and terminology suggested in the introduction.

1. In the original approach of Smirnov [28], the *δ-dimension* of a proximity space (X, δ) corresponds to the case, $\mathcal{C}$-dim, where $\mathcal{C}$ is the family of all δ-uniform covers. In fact, $\mathcal{C}$ is a basis of the totally bounded uniformity induced by proximity δ (see, for example, [10, Ch. 8.4]).

The *uniform dimension* δd_μ of Isbell [17] for uniform space (X, μ) corresponds to the case $\mathcal{C}$-dim, where $\mathcal{C}$ is the family of all finite covers $u \in \mu$. Therefore, $\mathcal{C}$ is a basis of the *precompact reflection* of μ (that is the totally bounded uniformity which is the supremum of all totally bounded uniformities less or equal to μ) [17, Ch. 2]. For all uniformities μ on X for which precompact reflections are the same, the covering dimension invariants δd_μ coincide.

For any space X there is a one-to-one correspondence between proximities and totally bounded uniformities. Hence, the collections of δ-dimensions and uniform dimensions coincide. We put

$$\text{d-Sp}_U(X) = \{\delta d_\mu X\colon\ \mu \text{ is the uniformity on } X\}.$$

From the General Theorem and the one-to-one correspondence between totally bounded uniformities and compactifications, it follows that

$$\text{d-Sp}_U(X) = \{\dim bX\colon\ bX \text{ is the compactification of } X\}.$$

This equality may be revised. For a uniform space (X, μ), let δ be the proximity corresponding to the precompact reflection of μ and $b_\mu X$ the Samuel compactification of X with respect to μ. Then,

$$\delta d_\mu X = \delta\text{-dimension}\, X = \dim b_\mu X.$$

2. Let $\omega X = \omega(X, \mathcal{F})$ be the Wallman-type compactification of a space X with respect to the normal base $\mathcal{F}$ (see, for example, [2]). Then, $\mathcal{C}$, the family of all finite covers by sets from $\mathcal{F}^c$, satisfies conditions (A) and (B) of the General Theorem. For the proof of this fact it is enough to apply Theorem 2.2 (d) from [2] (which states that any element from $\mathcal{C}$ has an extension on ωX), and Lemmas 2.2, 2.3 from [13] (asserting that any finite open cover of ωX has a shrinking which is an extension of some element from $\mathcal{C}$).

From this observation it follows that the *dimension* $\mathrm{d}(X, \mathcal{F})$ *by a normal base* $\mathcal{F}$ of a space X introduced by Iliadis in [15] (see, also, [13]) corresponds to the case $\mathcal{C}$-dim where $\mathcal{C}$ is the family of all finite covers by sets from $\mathcal{F}^c$. As it is noted in the introduction, $\mathcal{C}$ is the basis of some uniformity μ. The Samuel compactification of X with respect to μ is ωX. Hence,

$$\mathrm{d}(X, \mathcal{F}) = \delta d_\mu X = \dim \omega X.$$

Different normal bases on topological space X allow to introduce the variety of covering dimension invariants:

$$\text{d-Sp}_W(X) = \bigl\{\mathrm{d}(X, \mathcal{F})\colon\ \mathcal{F} \text{ is the normal base on } X\bigr\}$$

(let us remind that if for normal bases $\mathcal{F}$ and $\mathcal{G}$ on X the Wallman-type compactifications are the same (it is the case, where $\mathcal{F}$ separates $\mathcal{G}$ [23] and vice versa) then $\mathrm{d}(X, \mathcal{F}) = \mathrm{d}(X, \mathcal{G})$) which obviously coincide with the set

$$\{\dim \omega X\colon\ \omega X \text{ is the Wallman-type compactification of } X\}.$$

V. Uljanov [30] showed that not all compactifications are of the Wallman-type.

3. There is a one-to-one correspondence between complete rings of bounded functions $\mathcal{R}$ on X (they are also called uniformly closed rings of bounded functions) and compactifications of X (see, for example, [10, Problem 3.12.22]). Zero-sets $Z(\mathcal{R})$ of functions from $\mathcal{R}$ is a normal base on X which is a separating, nest-generated intersection ring of subsets [26]. Among the rings which generate the same family $Z(\mathcal{R})$ there is the largest one $\mathcal{R}_{\max}$. The ring $\mathcal{R}_{\max}$ will be called the *z-supremum of* $\mathcal{R}$. The ring $\mathcal{R}_{\max}$ is precisely the ring of functions on X which can be extended on the Wallman-type compactification $\omega(X, Z(\mathcal{R}))$ of X with respect to the normal base $Z(\mathcal{R})$. This compactification is the maximal among compactifications determined by rings with z-supremum $\mathcal{R}_{\max}$ (see, for example, [14,26]). It is called a β-like compactification [19]. Denote by $C^*(X, Y)$ the uniformly closed ring of functions on X which are restrictions on X of all functions from $C^*(Y)$. The β-like compactification ωX will be called the *z-supremum of compactification* bX if $\omega X = \omega(X, Z(C^*(X, bX)))$. It is equivalent that $C^*(X, \omega X)$ is the z-supremum of $C^*(X, bX)$.

For a uniformly closed ring $\mathcal{R}$ of bounded functions on a space X it is possible to define *dimension* $\mathrm{d}(X, \mathcal{R})$ *by a ring* $\mathcal{R}$ of X as $\mathcal{C}$-dim where $\mathcal{C}$ is the family of all finite covers by sets from $Z(\mathcal{R})^c$. As in item 2, the family $\mathcal{C}$ is the basis of some uniformity μ such that the Samuel compactification of X with respect to μ is $\omega(X, Z(\mathcal{R}))$. Hence,

$$\mathrm{d}(X, \mathcal{R}) = \mathrm{d}\bigl(X, Z(\mathcal{R})\bigr) = \delta d_\mu X = \dim \omega\bigl(X, Z(\mathcal{R})\bigr).$$

We put

$$\text{d-Sp}_\beta(X) = \bigl\{\mathrm{d}(X, \mathcal{R})\colon\ \mathcal{R} \text{ is a uniformly closed ring of bounded functions on } X\bigr\}.$$

The set $\text{d-Sp}_\beta(X)$ coincides with the set

$$\{\dim b_\beta X\colon\ b_\beta X \text{ is a } \beta\text{-like compactification of } X\}.$$

Remark 2.1. In order to study dimensions by rings of a space X it is enough to examine only the largest rings of functions on X which are exactly those that are extended on the corresponding β-like compactifications of X. Characterization of such rings can be found in [16,19,26].

For a uniform space (X, μ) the preimage of an open set under the uniform function $f : X \to I$ is called uniformly open [4]. The *uniform dimension* μ-dim of Charalambous of a uniform space (X, μ) [4] (see, also, [5]) corresponds to the case $\mathcal{C}$-dim where $\mathcal{C}$ is all finite covers of uniformly open sets.

For a space Y let $CZ(Y)$ (respectively, $Z(Y)$) be the set of all cozero-sets (respectively, zero-sets) of Y and for $X \subset Y$ we put $CZ(X, Y) = X \wedge CZ(Y)$ and $Z(X, Y) = X \wedge Z(Y)$. The *relative dimension* $\mathrm{d}(X, Y)$ of Chigogidze of $X \subset Y$ [8] corresponds to the case $\mathcal{C}$-dim where $\mathcal{C}$ is all finite covers by sets from $CZ(X, Y)$.

Theorem 2.2. *The following statements are true*:

(a) *For a uniform space (X, μ),*

$$\mu\text{-}\dim X = \mathrm{d}(X, \mathcal{R}_\mu),$$

where $\mathcal{R}_\mu$ is the ring of all bounded uniformly continuous functions on X.

(b) *For a subset X of a space Y,*

$$\mathrm{d}(X, Y) = \mathrm{d}\big(X, C^*(X, Y)\big).$$

(c) *For a space X*

$$\mathrm{d}\text{-}\mathrm{Sp}_\beta(X) = \{\mu\text{-}\dim X\colon\ \mu \textit{ is the uniformity on } X\} = \big\{\mathrm{d}(X, Y)\colon\ X \subset Y\big\}.$$

Proof. (a), (b) It is easy to verify, that for a uniformity μ on X $(X \subset Y)$ the set of all bounded uniformly continuous functions $\mathcal{R}_\mu$ (the family $C^*(X, Y)$) is a uniformly closed ring of bounded functions. The equality $\mu\text{-}\dim X = \mathrm{d}(X, \mathcal{R}_\mu)$ $(\mathrm{d}(X, Y) = \mathrm{d}(X, C^*(X, Y)))$ immediately follows from definition.

(c) The rings of bounded uniformly continuous functions on two uniform spaces (X, μ) and (X, μ') are the same iff the precompact reflections of μ and μ' coincide. Since there is a one-to-one correspondence between totally bounded uniformities and uniformly closed rings of bounded functions, we have

$$\mathrm{d}\text{-}\mathrm{Sp}_\beta(X) = \{\mu\text{-}\dim X\colon\ \mu \text{ is the uniformity on } X\}.$$

Since $C^*(X, Y) = C^*(X, \mathrm{cl}_{\beta Y} X)$ for $X \subset Y$, and

$$C^*(X, Y) = C^*\big(X, Y'\big) \quad \text{if and only if} \quad C^*(X, \mathrm{cl}_{\beta Y} X) = C^*(X, \mathrm{cl}_{\beta Y'} X)$$

there is a one-to-one correspondence between uniformly closed rings of bounded functions on X and different rings of the form $C^*(X, Y)$ for $X \subset Y$. Hence,

$$\mathrm{d}\text{-}\mathrm{Sp}_\beta(X) = \big\{\mathrm{d}(X, Y)\colon\ X \subset Y\big\}. \qquad \square$$

Remark 2.3. In [12] the notion of a (multiplicative) perfectly normal base was introduced and the question was put whether each multiplicative perfectly normal base on a space X arises from zero-sets or, precisely, if it is a separating, nest-generated intersection ring. In [7] it is shown that this is true for multiplicative normal bases on a separable metrizable space. In [14] it is shown that every multiplicative normal base $\mathcal{F}$ with the property that any $O \in \mathcal{F}^c$ is a countable union of elements from $\mathcal{F}$, is a separating, nest-generated intersection ring. From Propositions 1 and 2 of [12] it follows that such normal bases coincide with multiplicative perfectly normal bases. Thus, every multiplicative perfectly normal base on a space X is a separating, nest-generated intersection ring. Hence,

$$\mathrm{d}\text{-}\mathrm{Sp}_\beta(X) = \big\{\mathrm{d}(X, \mathcal{F})\colon\ \mathcal{F} \text{ is the multiplicative perfectly normal base on } X\big\}.$$

4. The classical dimension dim which is due to Čech for normal and to Katetov for Tychonoff spaces is the special case of uniform dimensions δd where the finest uniformity μ on X is considered. Hence,

$$\dim X = \mathrm{d}\big(X, C^*(X)\big) = \mathrm{d}\big(X, Z(X)\big) = \delta d_\mu(X).$$

Relations between covering dimension invariants of a space X and covering dimensions dim of its compactifications show that the investigation of the collections

$$\{\dim X\}, \quad \mathrm{d}\text{-}\mathrm{Sp}_\beta(X), \quad \mathrm{d}\text{-}\mathrm{Sp}_W(X), \quad \mathrm{d}\text{-}\mathrm{Sp}_U(X)$$

is closely connected with the investigation of dimensional properties of compactifications of X.

3. Relations between covering dimension invariants

Theorem 3.1. *For any space X we have*:

(a) $\dim X \in \mathrm{d}\text{-}\mathrm{Sp}_\beta(X) \subset \mathrm{d}\text{-}\mathrm{Sp}_W(X) \subset \mathrm{d}\text{-}\mathrm{Sp}_U(X)$;

(b) *if X is a pseudocompact space, then*

$$\text{d-Sp}_\beta(X) = \text{d-Sp}_W(X) = \text{d-Sp}_U(X);$$

(c) (*CH*) *if X is a separable space, then*

$$\text{d-Sp}_W(X) = \text{d-Sp}_U(X);$$

(d) *if X is a finite-dimensional separable metrizable non-compact space, then*

$$\{\dim X\} = \text{d-Sp}_\beta(X) \neq \text{d-Sp}_W(X).$$

Proof. The relations in (a) follow from observations in Section 2.

The equalities in (b) follow from the fact that every compactification of a Tychonoff space X is a β-like compactification if and only if X is pseudocompact [19].

The equality in (c) follows from the fact that under CH every compactification of a separable Tychonoff space is a Wallman-type compactification [3].

The equality in (d) follows from [26, Lemma 2.7]. To prove inequality in (d) it is enough to show that X has a Wallman-type compactification of dimension $\geqslant n = \dim X + 1$. We shall use the graph closure compactifications [25]. Let bX be a metrizable compactification of X. For a point $x \in bX \setminus X$, bX is the Alexandroff one-point compactification of $X' = bX \setminus \{x\}$. Let D be a countable discrete subset of X' and $f : D \to I^n$ any mapping with a dense image in I^n. The mapping f can be extended to the mapping $F : X' \to I^n$. The closure of the subset $\{(t, F(t)):\ t \in X'\}$ in the product $bX \times I^n$ is a metrizable compactum, containing $\{x\} \times I^n$, and X is homeomorphic with the graph $\{(t, F(t)):\ t \in X\}$ of $F|_X$. Thus, a metrizable compactification of X of dimension $\geqslant n$ is obtained. From [24] or [1] it follows that any metrizable compactification is a Wallman-type compactification. Hence, (d) is proved. □

Question 3.2.

(1) Is there a space X such that $\text{d-Sp}_W(X) \neq \text{d-Sp}_U(X)$?
(2) For what spaces X, $\text{d-Sp}_\beta(X) = \text{d-Sp}_U(X)$?
(3) For what spaces X, $\text{d-Sp}_\beta(X) = \text{d-Sp}_W(X)$?

Theorem 3.3. *For a space X and its compactification bX there is a β-like compactification $b_\beta X$ such that $b_\beta X \geqslant bX$ and $\dim b_\beta X \leqslant \dim bX$.*

Proof. If $\dim bX = \infty$, then βX is the required β-like compactification. Let $\dim bX < \infty$. Then,

$$\dim bX \geqslant \mathrm{d}(X, bX) = \dim \omega\big(X, Z(X, bX)\big)$$

[9, Theorems 2.13 and 2.5] and

$$\omega\big(X, Z(X, bX)\big) \geqslant bX$$

[26, Lemma 2.8]. Hence, $b_\beta X = \omega(X, Z(X, bX))$ is the required β-like compactification of X. □

Corollary 3.4. *For every space X we have*:

(a) *If* $\text{d-Sp}_U(X) \neq \{\infty\}$, *then*

$$\text{d-Sp}_W(X) \neq \{\infty\} \quad \textit{and} \quad \text{d-Sp}_\beta(X) \neq \{\infty\}.$$

(b) *If* $\text{d-Sp}_\beta(X) = \{\infty\}$, *then* $\text{d-Sp}_W(X) = \{\infty\}$ *and* $\text{d-Sp}_U(X) = \{\infty\}$.

The following corollary follows from the proof of Theorem 3.3.

Corollary 3.5. *Let $\{b_\alpha X:\ \alpha \in A\}$ be the family of compactifications of a space X with the same z-supremum $b_\beta X$. Then, $b_\beta X$ is a β-like compactification of X, $b_\beta X \geqslant b_\alpha X$ and $\dim b_\beta X \leqslant \dim b_\alpha X$, $\alpha \in A$.*

The dimension invariant $\operatorname{mindim} X$ is defined in [17] as

$$\operatorname{mindim} X = \min\big\{k:\ k \in \text{d-Sp}_U(X)\big\}.$$

Evidently, from Theorem 3.1 (a), it follows that

$$\min\big\{k:\ k \in \text{d-Sp}_\beta(X)\big\} \geqslant \min\big\{k:\ k \in \text{d-Sp}_W(X)\big\} \geqslant \min\big\{k:\ k \in \text{d-Sp}_U(X)\big\}.$$

Thus, from Theorem 3.3 we have the following theorem.

Theorem 3.6. *For a space X,*

$$\operatorname{mindim} X = \min\{k\colon\ k \in \text{d-Sp}_\beta(X)\} = \min\{k\colon\ k \in \text{d-Sp}_W(X)\} = \min\{k\colon\ k \in \text{d-Sp}_U(X)\}.$$

Remark 3.7.

(a) For a space X, $\operatorname{mindim} X = 0$ iff $\operatorname{ind} X = 0$ [17, Ch. VI, 3].
(b) For a locally compact space X, $\operatorname{mindim} X = \sup\{\dim K\colon\ K \subset X,\ K \text{ is compact}\}$ [17, Ch. VI, 15].
(c) For a Lindelöff space X, βX is the only β-like compactification of X (see, for example, [26, Lemma 2.7]) and, hence, $\operatorname{mindim} X = \dim X$ [17, Ch. VI, 23].

Example 3.8. (1) A Roy metrizable space M with $\dim M = 1$ and $\operatorname{ind} M = 0$ [20] is an example of a space with

$$\dim M = 1 > \operatorname{mindim} M = 0.$$

For any $n \in \mathbb{N} \cup \{\infty\}$, normal zero-dimensional spaces X_n with $\dim X_n = n$ [27] make the gap between $\dim X_n$ and $\operatorname{mindim} X_n$ to equal n.

(2) Let $X_{kn} = X_k \oplus Y_n$, where $k, n \in \mathbb{N}$, X_k is the space from (1), and Y_n is a compact space with $\dim Y_n = 1$ and $\operatorname{ind} Y_n = n$ (see, for example, [31]). Then, $\operatorname{ind} X_{kn} = n$, $\dim X_{kn} = k$, and $\operatorname{mindim} X_{kn} = 1$. Hence, there are spaces X and X' such that both double inequalities are possible:

$$\operatorname{ind} X > \dim X > \operatorname{mindim} X, \qquad \dim X' > \operatorname{ind} X' > \operatorname{mindim} X'.$$

Question 3.9.

(1) For what spaces X, $\dim X = \operatorname{mindim} X$?
(2) For what spaces X, $\operatorname{ind} X = \operatorname{mindim} X$?
(3) Is there a space X for which $\dim X > \operatorname{mindim} X > \operatorname{ind} X$?

4. Ranges of covering dimension invariants

For a nonnegative integer k we put $[k, \infty] = \{n \in \omega\colon\ n \geqslant k\} \cup \{\infty\}$, and for $k = \infty$, $[k, \infty] = \{\infty\}$.

Definition 4.1. Let bX be a compactification of X with $\dim bX = k \neq \{\infty\}$. A family

$$\{b_n X\colon\ n \in \omega \cup \{\infty\},\ n \geqslant k\}$$

of compactifications of X with $\dim b_n X = n$, such that

$$bX = b_k X \geqslant b_{k+1} X \geqslant \cdots; \quad bX \geqslant b_\infty X\ (bX \geqslant b_\infty X \geqslant \cdots \geqslant b_{k+1} X \geqslant b_k X)$$

is called an *increasing* (*decreasing*) *chain of compactifications beginning from* bX.

An in(de)creasing chain of compactifications beginning from bX which z-suprema coincide, is called an *in*(*de*)*creasing z-chain of compactifications beginning from* bX.

Theorem 4.2. *Suppose that the remainder $bX \setminus X$ of a finite-dimensional compactification bX of a space X contains a compact subset K which can be mapped onto $I = [0, 1]$. Then, there exists an in*(*de*)*creasing chain of compactifications beginning from bX. Hence,*

$$[\dim bX, \infty] \subset \text{d-Sp}_U(X).$$

If, moreover, K is a G_δ-subset, then there exist an in(*de*)*creasing z-chain of compactifications beginning from bX whose z-suprema coincide with z-supremum of bX.*

Proof. Since the set $X' = bX \setminus K$ is locally compact, for any continuous image K' of K, the space X' has a compactification whose remainder is K' (see for example [10, Theorem 3.5.13]). This compactification is also a compactification of X.

Let $b'X$ be any such compactification of X and K' the corresponding continuous image of K. By Dowker's theorem [11, Problem 3.1.B.(b)],

$$\dim b'X \leqslant \max\{\dim K', \operatorname{rd}(b'X \setminus K')\},$$

where

$$\operatorname{rd}(b'X \setminus K') = \sup\{\dim F\colon\ F \subset b'X \setminus K',\ F \text{ is closed in } b'X\}.$$

Closed subsets of $b'X$ are compact. Since $b'X \setminus K' = bX \setminus K$, their compact subsets coincide. Therefore,

$$\operatorname{rd}(b'X \setminus K') = \operatorname{rd}(bX \setminus K) \leqslant \dim bX \quad \text{and} \quad \dim b'X \geqslant \dim K'.$$

Hence,

$$\dim b'X = \dim K' \quad \text{for any } K' \text{ with } \dim K' \geqslant \dim bX.$$

Any Peano space (that is, compact, connected, locally connected metric space) is a continuous image of K. The required increasing chain of compactifications beginning from bX is constructed by induction. Let $\dim bX = k \geqslant 0$. The compactification $b_{k+1}X$ is a continuous image of $b_kX = bX$ under a map f_{k+1} such that f_{k+1} is identical on $bX \setminus K$ and $f_{k+1}(K) = I^{k+1}$. For $n \in \omega$, $n > k+1$, the compactification b_nX is a continuous image of $b_{n-1}X$ under a map f_n such that f_n is identical on $bX \setminus I^{n-1}$ and $f_n(I^{n-1}) = I^n$. Compactification $b_\infty X$ is a continuous image of bX under a map f_∞ such that f_∞ is identical on $bX \setminus K$ and $f_\infty(K) = I^\infty$.

The fulfillment of inclusion $[\dim bX, \infty] \subset \text{d-Sp}_U(X)$ is evident.

In order to prove the second statement of the theorem it suffices to show that

$$Z(X, bX) = Z(X, b_nX), \quad n \in \omega \cup \{\infty\},\ n \geqslant k.$$

Since, $bX \geqslant b_nX$, the inclusion $Z(X, bX) \supset Z(X, b_nX)$ is evident.

Since in a normal space any closed G_δ-subset is a zero-set, there is a function $f : bX \to I$ such that $K = f^{-1}(0)$. For any function $g : bX \to I$, we can consider the function $g_f = f \cdot g$. Then, we have

$$g_f^{-1}(0) \cap (bX \setminus K) = g^{-1}(0) \cap (bX \setminus K) \quad \text{and} \quad K \subset g_f^{-1}(0).$$

Hence, a function h may be defined on b_nX such that $h \circ \text{pr}_n = g_f$, where $\text{pr}_n : bX \to b_nX$ is a natural map of compactifications. For this function we have

$$h^{-1}(0) \cap \bigl(b_nX \setminus \text{pr}_n(K)\bigr) = g_f^{-1}(0) \cap (bX \setminus K).$$

This equality proves the inclusion

$$Z(X, bX) \subset Z(X, b_nX).$$

The required decreasing chain of compactifications beginning in bX is constructed taking as $b_\infty X$ a continuous image of bX under a map g_∞ such that g_∞ is identical on $bX \setminus K$ and $g_\infty(K) = I^\infty$. We can choose faces of cube I^∞ in such a way that I^n is a face of I^{n+1} and projections $p_n : I^\infty \to I^n$ satisfy condition $p_n = p_{n+1,n} \circ p_{n+1}$, $n \in \omega$ ($p_{n+1,n}$ is a natural projection of I^{n+1} onto I^n). The compactification b_nX is a continuous image of $b_\infty X$ under a map g_n such that g_n is identical on $b_\infty X \setminus I^\infty$ and $g_n(I^\infty) = I^n$, $n \in \omega$, $n \geqslant k$. The rest of the proof is the same as above. □

Corollary 4.3. *The following statements are true*:

(a) *For every finite-dimensional β-like compactification $b_\beta X$ of a space X either there is a z-chain of compactifications beginning from $b_\beta X$ or the set of compactifications with z-suprema $b_\beta X$ is $\{b_\beta X\}$.*

(b) *If a space X is not G_δ-dense in a finite-dimensional β-like compactification $b_\beta X$, then $[\dim b_\beta X, \infty] \subset \text{d-Sp}_U(X)$.*

(c) *If X is not a pseudocompact space, then $[\dim X, \infty] \subset \text{d-Sp}_U(X)$. If X is a Lindelöff non-compact space, then $\text{d-Sp}_U(X) = [\dim X, \infty]$. Moreover, if $\dim X = \infty$, then $\text{d-Sp}_U(X) = \text{d-Sp}_W(X) = \text{d-Sp}_\beta(X) = \{\infty\}$.*

(d) *If among β-like compactifications $b_\beta X$ of a space X with $\dim b_\beta X = \text{mindim}\, X$ or $\dim b_\beta X = \text{mindim}\, X + 1$ there is one, the remainder of which contains a compact subset K which can be mapped onto I, then $\text{d-Sp}_U(X) = [\text{mindim}\, X, \infty]$.*

(e) *If X is G_δ-dense in a β-like compactification $b_\beta X$ of X and the remainder $b_\beta X \setminus X$ contains a compact subset K which can be mapped onto I, then $[\dim b_\beta X, \infty] \subset \text{d-Sp}_\beta(X)$.*

(f) *If among β-like compactifications $b_\beta X$ of X with $\dim b_\beta X = \text{mindim}\, X$ or $\dim b_\beta X = \text{mindim}\, X + 1$ there is one, the remainder of which contains a compact subset K which can be mapped onto I and in which X is G_δ-dense, then $\text{d-Sp}_U(X) = \text{d-Sp}_\beta(X) = [\text{mindim}\, X, \infty]$.*

(g) *If there are β-like compactification $b_\beta X$ and compactification bX of X such that $b_\beta X \geqslant bX$ and $\dim b_\beta X > \dim bX$, then $[\dim b_\beta X, \infty] \subset \text{d-Sp}_U(X)$. Moreover, if X is G_δ-dense in $b_\beta X$, then $[\dim b_\beta X, \infty] \subset \text{d-Sp}_\beta(X)$.*

Proof. (a) If $b_\beta X$ is a β-like compactification of X, then either there is a compact G_δ-subset F in $b_\beta X \setminus X$ which contains $\mathbb{N}^*$ or X is G_δ-dense in $b_\beta X$ (see, for example, [19, §3]). Surjection of $\mathbb{N}$ on rationals $\mathbb{Q} \subset I$ can be extended to the surjection of $\beta\mathbb{N}$ on I. The image of $\mathbb{N}^*$ is dense in I and compact. Hence, I is a continuous image of $\mathbb{N}^*$ and thus of F. It remains to apply Theorem 4.2 to construct a z-chain of compactifications beginning from $b_\beta X$. If X is G_δ-dense in $b_\beta X$ then X is G_δ-dense in any compactification bX such that $bX < b_\beta X$ and, thus, bX is a β-like compactification of X [19, §3]. Hence, z-supremum of bX is $bX \neq b_\beta X$, and there is no compactifications of X with z-supremum $b_\beta X$ and which differs from $\{b_\beta X\}$.

(b) Immediately follows from (a) and Theorem 4.2.

(c) If X is not a pseudocompact space, then X is not G_δ-dense in βX (see, for example, [32, Theorem 3.6]). It remains to apply (a). The second statement follows from Remark 3.7 (c) and Corollary 3.4 (b).

(d) Follows from Theorem 4.2.

(e) Follows from (a) and Theorem 4.2.

(f) Follows from (e).

(g) From Hurewicz theorem on dimension-lowering maps (see, for example, [11, Theorem 3.3.10]) it follows that if $\dim b_\beta X > \dim bX$ then $b_\beta X \setminus X$ contains a compact subset of positive dimension (as an inverse image of some point) and, hence, β-like compactification $b_\beta X$ contains a compact set which can be mapped onto I. The rest follows from (e). □

Recall [10, Problem 1.7.10] that any topological space is the disjoint union of a perfect closed set and a scattered set. It is easy to prove that any nonempty perfect compact space F can be mapped onto I.

Theorem 4.4. *Let X be a locally compact space. The following statements are true*:

(a) *If $X^* = \beta X \setminus X$ contains a nonempty perfect set, then*

$$\text{d-Sp}_U(X) = [\operatorname{mindim} X, \infty].$$

(b) *If X^* is a scattered space, then X is pseudocompact and*

$$\text{d-Sp}_U(X) = \text{d-Sp}_\beta(X) = \{\dim X\}.$$

Proof. (a) By Remark 3.7 (b) $\operatorname{mindim} X = \max\{\dim K\colon\ K \subset X,\ \ K \text{ is compact}\}$. Put $\operatorname{mindim} X = k$. Evidently, for any compactification bX of X we have $\dim bX \geqslant k$ and $\dim bX \geqslant \dim(bX \setminus X)$. On the other hand,

$$\dim bX \leqslant \max\bigl\{\dim(bX \setminus X), k\bigr\}$$

since $k = \operatorname{rd}(bX \setminus (bX \setminus X))$. Hence, $\dim bX = k$ if $\dim(bX \setminus X) \leqslant k$ and $\dim bX = \dim(bX \setminus X)$ if $\dim(bX \setminus X) > k$. Since $\beta X \setminus X$ is compact and contains a compact space which can be mapped onto I, the dimensions of remainders of compactifications may be arbitrary integer $\geqslant 0$ or ∞.

(b) If X^* is a scattered space, then X is pseudocompact because, otherwise, X^* must contain $\mathbb{N}^*$, and, thus, a nonempty perfect subset. The continuous image of a scattered compact space is scattered [21] and, hence, all continuous images of X^* are zero-dimensional. Thus, for any compactification bX of X, $\dim bX = \dim X$. □

Corollary 4.5. *If X is a locally compact not pseudocompact space, then*

$$\text{d-Sp}_U(X) = [\operatorname{mindim} X, \infty].$$

Example 4.6. (1) There is a locally compact pseudocompact space X such that

$$\text{d-Sp}_\beta(X) = [0, \infty].$$

Indeed, let X be a Mrówka's countable locally compact pseudocompact space [10, Exercise 3.6.I.(a)] such that $\beta X \setminus X$ is an uncountable metrizable compact space K (see [29, Theorem 2.1]). Then, every compact subset of X is zero-dimensional [29, Lemma 1.1] and, hence,

$$\dim bX = \dim(bX \setminus X)$$

for any compactification bX of X. Since an uncountable metrizable compactum K contains a perfect set (by the Cantor–Bendixson theorem), the rest of the proof follows from Theorem 4.4.

(2) Countable ordinals is an example of a locally compact pseudocompact space X with

$$\text{d-Sp}_\beta(X) = \{0\}.$$

(3) There is a pseudocompact space X such that

$$\text{d-Sp}_\beta(X) = \{0, 1\}.$$

From Corollary 4.3 (g) it follows that for such a space X, $\dim \beta X = 0$ and there is a compactification bX of X with $\dim bX = \operatorname{mindim} X = 1$.

Let

$$X = (W_1 \times C) \setminus \bigl(\{\omega_1\} \times D\bigr),$$

where W_1 is the union of the set of all countable ordinals and the singleton $\{\omega_1\}$, $C \subset I$ is the Cantor set, and D is the countable set of end points of adjoint intervals. Evidently, $\beta X = W_1 \times C$ and the remainder X^* is countable. Let $\sim$ be the equivalence relation on βX such that two points (α, t) and (α', t') of $= W_1 \times C$ are equivalent if and only if $\alpha = \alpha'$ and $t = t'$ or $\alpha = \alpha' = \omega_1$ and t, t' are the end points of some adjoint interval to C. It is easy to check that $bX = \beta X/{\sim}$ is a compactification of X with $\dim bX = 1$ (bX contains a segment I).

In order to show that there are no compactifications of X with dimension different from $\{0, 1\}$ it is enough to prove that for any continuous map f of C for which $f(C \setminus D) \cap f(D) = \emptyset$ and $f|_{C\setminus D}$ is a homeomorphism, $\dim f(C) \leqslant 1$. The space $f(C)$ is compact metrizable (see, for example, [10, Theorem 4.4.15]) and, thus, $\dim f(C) = \operatorname{Ind} f(C)$. The image of a countable set D is countable and, thus, zero-dimensional. If $\dim f(C) > 1$, then there is a pair of disjoint closed subsets of $f(C)$ such that any partition between them is at least one-dimensional. By the separation theorem [11, Theorem 1.2.11] we can find a partition L which misses $f(C)$ and $\dim L > 0$. But L is a subset of C and, thus, zero-dimensional. This contradiction completes the proof.

Question 4.7.

(a) Does equality

$$\text{d-Sp}_U(X) = [\operatorname{mindim} X, \infty]$$

hold for a not pseudocompact space X?

(b) What can be said about $\text{d-Sp}_\beta(X)$ of a pseudocompact not locally compact space X?

Theorem 4.8. *If X is a finite-dimensional separable metrizable non-compact space, then*

$$\text{d-Sp}_W(X) = [\operatorname{mindim} X, \infty].$$

Proof. From Remark 3.7 (c) it follows that $\operatorname{mindim} X = \dim X$ and by the compactification theorem (see, for example, [11, Theorem 3.4.2]), X has a metrizable compactification bX with $\dim bX = \operatorname{mindim} X$. As in the proof of item (d) of Theorem 3.1, a metrizable compactification $b'X$ of X with $\dim b'X \leqslant \dim bX + 1$ containing segment I can be constructed. By Corollary 4.3 (d), $\text{d-Sp}_U(X) = [\operatorname{mindim} X, \infty]$. Moreover, all compactifications on which rangers of $\text{d-Sp}_U(X)$ are realized, are metrizable spaces. It remains to remind that metrizable compactifications of a space are Wallman-type compactifications. $\square$

Problem 4.9.

(1) Characterize those subsets of the set $\omega \cup \{\infty\}$ which can be $\text{d-Sp}_W(X)$ or $\text{d-Sp}_\beta(X)$ for some space X.
(2) Characterize those spaces X which have as $\text{d-Sp}_W(X)$ or $\text{d-Sp}_\beta(X)$ a given subset of $\omega \cup \{\infty\}$.

Some concrete questions which arise from Problem 4.9 may be interesting.

Question 4.10.

(a) Is there a not pseudocompact space X with

$$\text{d-Sp}_W(X) = \{\dim X\} \neq \{\infty\}?$$

(b) Is there a not pseudocompact space X such that $\{\infty\} \notin \text{d-Sp}_W(X)$?
(c) Is there a space X with $\text{d-Sp}_W(X) = \{1, 2, 3\}$ (respectively, $\text{d-Sp}_\beta(X) = \{1, 2, 3\}$)?

5. Some properties of covering dimension invariants

Theorem 5.1. *For every normal base $\mathcal{G}$ on a space X and any compactification bX of X such that $bX \leqslant \omega(X, \mathcal{G})$ there exists a normal base $\mathcal{F}$ on X contained in $\mathcal{G}$ such that $bX \leqslant \omega(X, \mathcal{F}) \leqslant \omega(X, \mathcal{G})$, $|\mathcal{F}| = w(bX)$ and $d(X, \mathcal{F}) \leqslant d(X, \mathcal{G})$.*

Proof. Let $\mathrm{d}(X, \mathcal{G}) = n$, $n \in \omega \cup \{\infty\}$, and $w(bX) = \tau$. Without lost of generality we can assume that τ is an infinite cardinal. For every $k \in \omega$ we shall construct an indexed family $\mathcal{F}_k = \{F_k^\delta\colon\ \delta \in \tau\}$ such that the family $\mathcal{F} = \bigcup\{\mathcal{F}_k\colon\ k \in \omega\}$ will be the required normal base on X which is contained in $\mathcal{G}$. The construction of these indexed families will be done by induction on k.

Let $f : \omega(X, \mathcal{G}) \to bX$ be the natural map of compactifications. Since f is perfect, $f(\omega\mathcal{G}) = \{f(\omega F)\colon\ F \in \mathcal{G}\}$ is a base for closed sets on bX and we can choose a base $\mathcal{F}'_{-1}$ of $|\mathcal{F}'_{-1}| \leqslant \tau$ which is contained in $f(\omega\mathcal{G})$. Put $\mathcal{F}_{-1} = \{F \in \mathcal{G}\colon f(\omega F) \in \mathcal{F}'_{-1}\}$. It is easy to see that $\mathcal{F}_{-1}$ is a base on X.

A pair (G, K) of elements of $\mathcal{F}_{-1}$ is called *$\mathcal{G}$-separated* if there exists an element $F \in \mathcal{G}$ such that $G \subseteq X \setminus F \subseteq K$. In this case, F is called a *$\mathcal{G}$-separator* for (G, K).

For every $\mathcal{G}$-separated pair of elements of $\mathcal{F}_{-1}$ we choose a fixed $\mathcal{G}$-separator and denote by $\mathcal{F}_0$ the union of the set of all such $\mathcal{G}$-separators and the set $\mathcal{F}_{-1}$. Obviously, $\mathcal{F}_0$ is a base of cardinality $\leqslant \tau$ for the closed subsets of X. We prove that $\mathcal{F}_0$ is disjunctive.

Indeed, let Q be an element of $\mathcal{F}_0$ and x a point of $X \setminus Q$. Since $\mathcal{F}_{-1}$ is a base for the closed subsets of X there exists an element Q' of $\mathcal{F}_{-1}$ such that $Q \subseteq Q'$ and $x \notin Q'$. Since $\mathcal{G}$ is disjunctive there exists an element F' of $\mathcal{G}$ such that $x \in F'$ and $F' \cap Q' = \emptyset$. Since $\mathcal{G}$ is base-normal there exist elements L and H of $\mathcal{G}$ such that the pair (L, H) is a screening of the pair (F', Q'). Let H' be an element of $\mathcal{F}_{-1}$ such that $x \notin H'$ and $H \subseteq H'$. Therefore, we have $Q' \subseteq X \setminus L \subseteq H'$, which means that the pair (Q', H') is $\mathcal{G}$-separated. Let F be the chosen $\mathcal{G}$-separator of (Q', H'). Then, $x \in F$ and $Q \cap F = \emptyset$, which means that $\mathcal{F}_0$ is disjunctive.

Let $\mathcal{F}_0 = \{F_0^\delta\colon \delta \in \tau\}$. Suppose that the indexed families $\mathcal{F}_m$ have been constructed for all integers m, $0 \leqslant m < k$. We shall construct the indexed family $\mathcal{F}_k$. First we consider the family $\{F_m^\delta\colon (m,\delta) \in \{0,\dots,k-1\} \times \tau\}$ which is considered as an indexed family having the set $\{0,\dots,k-1\} \times \tau$ as the indexing set.

Below, for every set S we denote by $\mathrm{Fin}(S)$ the set of all nonempty finite subset of S and by $\mathrm{Fin}_p(S)$, $p \in \omega \setminus \{0\}$, the subset of all elements of $\mathrm{Fin}(S)$ consisting of p elements. Furthermore, we denote (a) by $f_\vee^k$ and $f_\wedge^k$ two one-to-one mappings of the set $\mathrm{Fin}(\{0,\dots,k-1\} \times \tau)$ into the set $\{k\} \times \tau$, (b) by f_{cov}^k a one-to-one mapping of the set $\mathrm{Fin}(\{0,\dots,k-1\} \times \tau)$ into the set $\mathrm{Fin}(\{k\} \times \tau)$ such that if $f_{cov}^k(x) = y$, where $x \in \mathrm{Fin}(\{0,\dots,k-1\} \times \tau)$ and $y \in \mathrm{Fin}(\{k\} \times \tau)$, then $|x| = |y|$, and (c) by f_2^k a one-to-one mapping of $\mathrm{Fin}_2(\{0,\dots,k-1\} \times \tau)$ into the set $\mathrm{Fin}_2(\{k\} \times \tau)$. Without loss of generality, we can assume that (a) the ranges of the mappings $f_\vee^k$ and $f_\wedge^k$ are disjoint, (b) any two different elements of the range of f_{cov}^k or of the range of f_2^k are disjoint, (c) any element of the range of f_{cov}^k or of the range of f_2^k does not intersect the union of ranges of $f_\vee^k$ and $f_\wedge^k$, and (d) any element of the range of f_{cov}^k does not intersect any element of the range of f_2^k.

Now, we construct the indexed family $\mathcal{F}_k$. Let (k,ε) be an element of $\{k\} \times \tau$. If there exists an element $x = \{(m_0,\delta_0),\dots,(m_p,\delta_p)\}$ of $\mathrm{Fin}(\{0,\dots,k-1\} \times \tau)$ such that $f_\vee^k(x) = (k,\varepsilon)$ (respectively, $f_\wedge^k(x) = (k,\varepsilon)$), then we set $F_k^\varepsilon = \bigcup\{F_{m_i}^{\delta_i}\colon i \in \{0,\dots,p\}\}$ (respectively, $F_k^\varepsilon = \bigcap\{F_{m_i}^{\delta_i}\colon i \in \{0,\dots,p\}\}$).

Let $x = \{(m_0,\delta_0),\dots,(m_p,\delta_p)\}$ be an element of $\mathrm{Fin}(\{0,\dots,k-1\} \times \tau)$ and $f_{cov}^k(x) = y$, where $y = \{(k,\varepsilon_0),\dots,(k,\varepsilon_p)\}$. If the set $\pi = \{X \setminus F_{m_0}^{\delta_0},\dots,X \setminus F_{m_p}^{\delta_p}\}$ is not a cover of X, then we put $F_k^{\varepsilon_i} = \emptyset$ for every $i \in \{0,\dots,p\}$. If the set π is a cover of X, then there exists an open cover $\pi' = \{U_0,\dots,U_p\}$ of X with the order $\leqslant n$ consisting of elements of $\mathcal{G}^c$ such that $U_i \subset X \setminus F_{m_i}^{\delta_i}$, $i \in \{0,\dots,p\}$. In this case we put $F_k^{\varepsilon_i} = X \setminus U_i$, $i \in \{0,\dots,p\}$.

Let $f_2^k(\{(m_0,\delta_0),(m_1,\delta_1)\}) = \{(k,\varepsilon_0),(k,\varepsilon_1)\}$. If the intersection of the sets $F_{m_0}^{\delta_0}$ and $F_{m_1}^{\delta_1}$ is not empty, then we put $F_k^{\varepsilon_0} = F_k^{\varepsilon_1} = \emptyset$. If the pair $(F_{m_0}^{\delta_0}, F_{m_1}^{\delta_1})$ consists of disjoint elements of $\mathcal{G}$, then there exists a screening (F_0, F_1) of this pair by elements of $\mathcal{G}$. In this case we put $F_k^{\varepsilon_0} = F_0$ and $F_k^{\varepsilon_1} = F_1$.

Finally, if an element (k,ε) of $\{k\} \times \tau$ does not belong to the ranges of the mappings $f_\vee^k$ and $f_\wedge^k$ and does not belong to any element of the ranges of the mappings f_{cov}^k and f_2^k, then we put $F_k^\varepsilon = \emptyset$. Thus, the indexed set $\mathcal{F}_k$ is constructed.

Let ψ be a one-to-one mapping of the set $\omega \times \tau$ onto the set τ. Then, the indexed set $\{F_\eta\colon \eta \in \tau\}$ where $F_\eta = F_m^\delta$ if $\eta = \psi(m,\delta)$, is an indexation of the set $\mathcal{F} = \bigcup\{\mathcal{F}_k\colon k \in \omega\}$. We prove that this indexed set is the required normal base contained in $\mathcal{G}$. Obviously, by the construction, $\mathcal{F} \subset \mathcal{G}$ and $|\mathcal{F}| = \tau$. Since the subfamily $\mathcal{F}_0$ of $\mathcal{F}$ is a disjunctive base for closed subsets of X, $\mathcal{F}$ is also a disjunctive base for closed subsets of X.

We prove that $\mathcal{F}$ is a ring. Let $F_{\eta_0},\dots,F_{\eta_p}$ be elements of $\mathcal{F}$ and $\psi^{-1}(\eta_i) = (m_i,\delta_i)$, $i \in \{0,\dots,p\}$. Let also $k = \max\{m_0,\dots,m_p\} + 1$. Then, the set $\{(m_0,\delta_0),\dots,(m_p,\delta_p)\}$ is an element of $\mathrm{Fin}(\{0,\dots,k-1\} \times \tau)$. Let $(k,\varepsilon) = f_\vee^k(\{(m_0,\delta_0),\dots,(m_p,\delta_p)\})$ and $\eta = \psi(k,\varepsilon)$. Then, by the construction, $F_\eta = F_k^\varepsilon = \bigcup\{F_{m_i}^{\delta_i}\colon i \in \{0,\dots,k-1\}\}$ proving that $\mathcal{F}$ is closed under finite unions. Similarly we prove that $\mathcal{F}$ is closed under finite intersections. Therefore, $\mathcal{F}$ is a ring.

We prove that $\mathcal{F}$ is base-normal. Let (F_{η_0}, F_{η_1}) be a pair of disjoint elements of $\mathcal{F}$. Let $\psi^{-1}(\eta_0) = (m_0,\delta_0)$, $\psi^{-1}(\eta_1) = (m_1,\delta_1)$, $k = \max\{m_0,m_1\} + 1$, $f_2^k(\{(m_0,\delta_0),(m_1,\delta_1)\}) = \{(k,\varepsilon_0),(k,\varepsilon_1)\}$, $\psi(k,\varepsilon_0) = \eta_0'$, and $\psi(k,\varepsilon_1) = \eta_1'$. Then, by the construction, the pair $(F_{\eta_0'}, F_{\eta_1'}) = (F_k^{\varepsilon_0}, F_k^{\varepsilon_1})$ is a screening of the pair $(F_{\eta_0}, F_{\eta_1}) = (F_{m_0}^{\delta_0}, F_{m_1}^{\delta_1})$ proving that $\mathcal{F}$ is base-normal.

Finally, we prove that $\mathrm{d}(X,\mathcal{F}) \leqslant n$. Let $\pi = \{X \setminus F_{\eta_0},\dots,X \setminus F_{\eta_p}\}$ be a finite cover of X by elements of $\mathcal{F}^c$. Let $\psi^{-1}(\eta_i) = (m_i,\delta_i)$, $i \in \{0,\dots,p\}$, $k = \max\{m_0,\dots,m_p\} + 1$, $f_{cov}^k(\{(m_0,\delta_0),\dots,(m_p,\delta_p)\}) = \{(k,\varepsilon_0),\dots,(k,\varepsilon_p)\}$, and $\psi(k,\varepsilon_i) = \eta_i'$ for every $i \in \{0,\dots,p\}$. Then, by the construction, $\pi' = \{X \setminus F_{\eta_0'},\dots,X \setminus F_{\eta_p'}\} = \{X \setminus F_k^{\varepsilon_0},\dots,X \setminus F_k^{\varepsilon_p}\}$ is a cover of X of the order $\leqslant n$ by elements of $\mathcal{F}$ which is a refinement of $\pi = \{X \setminus F_{m_0}^{\delta_0},\dots,X \setminus F_{m_p}^{\delta_p}\}$ proving that $\mathrm{d}(X,\mathcal{F}) \leqslant n$.

In order to prove that $bX \leqslant \omega(X,\mathcal{F})$ we shall use Theorem 3.2.1 from [10]. Let T_1, T_2 be disjoint closed subsets of bX. Since $\mathcal{F}'_{-1}$ is a base on bX and T_1, T_2 are compact, there are elements $T_j^i \in \mathcal{F}'_{-1}$, $j = 1,\dots,l$, $i = 1,2$, such that $T_i \subset T^i = \bigcap\{T_j^i\colon j = 1,\dots,l\}$, $i = 1,2$, and $T^1 \cap T^2 = \emptyset$. Moreover, by the construction $T^i \cap X \in \mathcal{F}$, $i = 1,2$, and $\omega(T^1,\mathcal{F}) \cap \omega(T^2,\mathcal{F}) = \emptyset$. Hence, the identity map of X into bX can be extended onto $\omega(X,\mathcal{F})$, which means that $bX \leqslant \omega(X,\mathcal{F})$. The order $\omega(X,\mathcal{F}) \leqslant \omega(X,\mathcal{G})$ follows from the fact that $\mathcal{F} \subset \mathcal{G}$. $\square$

Remark 5.2. Theorem 5.1 may be examined as a reformulation of Theorem 2 from [18] where the outline of the proof is presented. Here, we give its complete proof using another method of the construction of the required normal base $\mathcal{F}$ from $\mathcal{G}$, with indication of the mappings $f_\vee^k$, $f_\wedge^k$, f_{cov}^k, f_2^k and ψ. This method can be found in [2]. Our aim is to underline that these mappings are independent from the considered space. This independence plays an important role in the proof of Theorem 5.6 below, where Theorem 5.1 is used simultaneously for a collection of spaces.

The well-known compactification theorem of Skljarenko (see, for example, [11, Theorem 3.4.2]) may be obviously formulated in the following way.

For the Stone–Čech compactification βX of X there exists a compactification bX such that: (1) $bX \leqslant \beta X$; (2) $w(bX) \leqslant w(X)$; (3) $\dim bX \leqslant \dim \beta X$.

Since for any compactification bX of a space X there is a compactification $b'X$ such that $b'X \leqslant bX$ and $w(b'X) = w(X)$, the following results follow from Theorem 5.1 and are a generalization of Skljarenko's theorem on the Wallman-type compactifications.

Corollary 5.3. *The following statements are true*:

(a) *For every Wallman-type compactification ωX of X there exists a Wallman-type compactification $\omega' X$ of X such that $\omega' X \leqslant \omega X$, $w(\omega' X) = w(X)$ and $\dim \omega' X \leqslant \dim \omega X$.*
(b) *For every Tychonoff space X there exists a Wallman-type compactification ωX of X such that*

$$\dim \omega X = \operatorname{mindim} X \quad \text{and} \quad w(\omega X) = w(X).$$

Question 5.4. Does Corollary 5.3 (a) remain true if we replace the Wallman-type compactification ωX by an arbitrary compactification bX?

Theorem 5.5. *For any compactification bX of X there are Wallman-type compactifications ωX and $\omega' X$ such that*:

(a) $bX \leqslant \omega X \leqslant \omega(X, Z(X, bX))$, $\dim \omega X \leqslant \dim \omega(X, Z(X, bX)) \leqslant \dim bX$ *and* $w(\omega X) \leqslant w(bX)$;
(b) $\omega' X \leqslant \omega X$, $\dim \omega' X \leqslant \dim \omega X$ *and* $w(\omega' X) = w(X)$.

Proof. (a) Follows from Theorems 3.3 and 5.1. (b) Follows from Corollary 5.3 (a). □

Theorem 5.6. *Let τ be an infinite cardinal and $n \in \omega \cup \{-1, \infty\}$. In the class of all spaces X with $w(X) \leqslant \tau$ and* $\operatorname{mindim} X \leqslant n$ *there exist compact universal spaces.*

Proof. Let $\mathbb{P}$ be the considered class and $\mathbf{S}$ a collection of elements of $\mathbb{P}$ such that each element of $\mathbb{P}$ is homeomorphic to an element of $\mathbf{S}$. For every $X \in \mathbf{S}$ we take a normal base $\mathcal{G}^X$ on X such that $\mathrm{d}(X, \mathcal{G}^X) \leqslant n$. In the same manner as in Theorem 5.1 for every $X \in \mathbf{S}$ we construct an indexed normal subbase $\mathcal{F}^X = \{F_\eta\colon\ \eta \in \tau\}$ of $\mathcal{G}^X$ on X such that $|\mathcal{F}^X| \leqslant \tau$ and $\mathrm{d}(X, \mathcal{F}^X) \leqslant n$. Moreover, in this construction the corresponding mappings $f^k_\vee$, $f^k_\wedge$, f^k_{cov}, f^k_2, and ψ are considered to be independent of the space X.

Let $\mathbf{M}$ be the co-mark $\{\{F_\eta\colon\ \eta \in \tau\}\colon\ X \in \mathbf{S}\}$ of $\mathbf{S}$ and R an $\mathbf{M}$-admissible family of equivalence relations on $\mathbf{S}$. Then, similarly to the proof of Lemma 5.4.4 from [15] it can be proved that the base $\mathcal{F} = C^{\mathrm{T},\diamond}_\tau$ (see [15, Section 5.1]) for the closed subsets of the containing space $\mathrm{T} = \mathrm{T}(\mathbf{M}, \mathrm{R})$ is a normal base such that $\mathrm{d}(\mathrm{T}, \mathcal{F}) \leqslant n$. Therefore, $\operatorname{mindim}(\mathrm{T}) \leqslant n$.

By Corollary 5.2 (a) there exists a compactification $b(\mathrm{T})$ of T such that $w(\mathrm{T}) = \tau$ and $\operatorname{mindim}(b(\mathrm{T})) \leqslant n$. This compactification is a required compact universal space. □

In [17, VI, Exercise 2] it is noted that $\operatorname{mindim} X \times Y \leqslant \operatorname{mindim} X + \operatorname{mindim} Y$. A simple proof of this inequality is given below.

Theorem 5.7. *For any spaces X_1 and X_2 we have*

$$\operatorname{mindim}(X_1 \times X_2) \leqslant \operatorname{mindim} X_1 + \operatorname{mindim} X_2.$$

Proof. Let $\mathcal{F}_j$ be a normal base on X_j such that $\mathrm{d}(X_j, \mathcal{F}_j) = \operatorname{mindim} X_j$, $j = 1, 2$. Then, by the Product Theorem 2.22 from [13],

$$\operatorname{mindim} X_1 \times X_2 \leqslant \mathrm{d}(X_1 \times X_2, \mathcal{F}_1 \otimes \mathcal{F}_2) \leqslant \mathrm{d}(X_1, \mathcal{F}_1) + \mathrm{d}(X_2, \mathcal{F}_2) = \operatorname{mindim} X_1 + \operatorname{mindim} X_2. \qquad \square$$

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